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The Structure of Indices of Control Systems for Single – Delay Autonomous Scalar Differential Equations

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Abstract: **This paper derived the structure of the indices of control systems for a class of single – delay autonomous linear differential equations on any given interval of length equal to the delay h for non –negative time periods. The formulation and the development of the theorem exploited part of an existing result on the interval a compact interval. The derivation of the associated solution matrices exploited the continuity of these matrices for positive time periods, the method of steps and backward continuation recursions to obtain these matrices on successive intervals of length equal to the delay h. The proof was achieved using combinations of integrals, summation notations, change of variables technique integrals, as well as the method of steps to obtain these matrices on successive intervals of length equal to the delay h. The indices were derived using the stage – wise algorithmic format, starting from the right – most interval of length h. This theorem globally extends the time scope of applications of these matrices to the solutions of terminal function problems and rank conditions for controllability and cores of targets.**

Keywords: **Structure, Indices, control systems, Method of steps, Change of variables.**

1. INTRODUCTION

The importance of indices of control systems matrices arises from the fact that they not only pave the way for the derivation of determining matrices for the determination of Euclidean controllability and compactness of cores of Euclidean targets but can be used independently for such determination. In sharp contrast to determining matrices the use of indices of control systems for the investigation of the Euclidean controllability of systems can be quite computationally challenging; however this difficulty can be mitigated if the coefficient matrices associated with the state variables turn out to be scalars. This paper pioneers the development of the structure of these indices.

Literature on state space approach to control studies is replete with indices of control systems as key components for the investigation of controllability. See [1], [2], [3], [4], [5], and [6], [7], [8]. Regrettably up to the year 2013, no author had made any attempt to obtain general expressions for the associated matrices or special cases of such matrices involving the delay h. The usual approach has been to start from the interval $[t_1 - h, t_1]$ and compute the index matrices for given problem instances; then the method of steps and backward continuation recursive procedure are deployed to extend these to the intervals $[t_1 - (k+1)h,\ t_1 - kh],$ for positive integral k , not exceeding 2, for the most part. Such approach is rather restrictive and doomed to failure in terms of structure for arbitrary k . In other words such approach fails to address the issue of the structure of control index matrices. The need to address such short-comings has become imperative; this is the major contribution of this paper, in the case of the scalar counterparts, with its wide-ranging implications for extensions to systems and holistic approach to controllability studies.

2. MATERIALS AND METHODS

Consider the system:

$$\frac{\partial}{\partial \tau} X(\tau,t) = -X(\tau,t)A_0 - X(\tau+h,t)A_1 \tag{1}$$

for $0<\tau<t,\ \tau \neq t-kh,\ k=0,1,\ldots$ where:

$$X(\tau,t)=\begin{cases} I_n; & \tau=t \\ 0; & \tau>t \end{cases} \tag{2}$$

A_0, A_1 are $n\times n$ constant matrices and $\tau \to X(\tau,t), \tau \to X(\tau,t+h)$ are $n\times n$ matrix functions.

See [2], [9] and [5] for properties of $X(t,\tau)$. Of particular importance is the fact that $\tau \to X(\tau,t)$ is analytic on the intervals $(t_1-(j+1)h,\ t_1-jh),\ j=0,1,\ldots;\ t_1-(j+1)h>0$. Any such $\tau\in(t_1-(j+1)h,\ t_1-jh)$ is called a regular point of $\tau\to X(t,\tau)$.

2.1 Definition:

The expression $c^* X(\tau,t_1)B$ is called the index of a given control system, where c is an n - dimensional constant column vector, $X(\tau,t_1)$ is defined in (1), B is an $n\times m$ constant matrix associated with the control

system $\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t)$ and $u(.)$ is an m-vector admissible control function. Thus the index matrix,

$X(\tau,t_1)$ determines the structure of the index of a given control system.

$$\text{Let } K_j = \left[t_1-(j+1)h, t_1-jh\right], \forall\, j: t_1-(j+1)h>0, \text{ and fixed } t_1>0. \tag{3}$$

[1] obtained the following expression for the above solution matrices associated

with the index of control system on successive sub-intervals of $[t_1\text{-}4h, t_1]$ of length h.

$$X(\tau,t_1)=\begin{cases} e^{A_0(t_1-\tau)}, \tau\in K_0; & (4)\\ e^{A_0(t_1-\tau)} - \int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)}A_1 e^{A_0(s_1-\tau)}ds_1,\ \tau\in K_1; & (5)\\ e^{A_0(t_1-\tau)} - \int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)}A_1 e^{A_0(s_1-\tau)}ds_1 + \int_{t_1-2h}^{\tau}\int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)}A_1 e^{A_0(s_1-h-s_2)}A_1 e^{A_0(s_2-\tau)}ds_1 ds_2 & (6)\\ \text{for } \tau\in K_2; & \\ e^{A_0(t_1-\tau)} - \int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)}A_1 e^{A_0(s_1-\tau)}ds_1 + \int_{t_1-2h}^{\tau}\int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)}A_1 e^{A_0(s_1-h-s_2)}A_1 e^{A_0(s_2-\tau)}ds_1 ds_2, & \\ -\int_{t_1-3h}^{\tau}\int_{t_1-2h}^{s_3+h}\int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)}A_1 e^{A_0(s_1-h-s_2)}A_1 e^{A_0(s_2-h-s_3)}A_1 e^{A_0(s_3-\tau)}ds_1 ds_2 ds_3, \tau\in K_3. & (7)\end{cases}$$

He also interrogated some topological dispositions of the index matrices and deduced that the index matrices are continuous on the interval $[t_1-4h, t_1]$ but not analytic there due to the break- down of analyticity for $\tau\in\{t_1,\ t_1-h,\ t_1-2h,\ t_1-3h\}$. These results are consistent with the existing qualitative theory on $X(\tau,t)$.

See [2], [5] and [6], [7], [8]. See also [10] for discussions on Analytical Functions and Topological Spaces.

The aim of this article is to generalize this result to arbitrary nonnegative sub – intervals of length h with respect to the scalar counterpart. In the sequel set $A_0 = a$ and $A_1 = b$, in (1). Then (4) through (7) yield:

$$X(\tau,\, t_1) = \begin{cases} S_0 = e^{a(t_1-\tau)}, \tau \in K_0; & (8) \\ S_1 = e^{a(t_1-\tau)} - b\displaystyle\int_{t_1-h}^{\tau} e^{A_0(t_1-\tau-h)} ds_1 = e^{a(t_1-\tau)} + b\left(t_1-\tau-h\right)e^{a(t_1-\tau-h)},\ \tau \in K_1; & (9) \\ S_2 = S_1 + \displaystyle\int_{t_1-2h}^{\tau}\int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-h-s_2)} A_1 e^{A_0(s_2-\tau)} ds_1 ds_2 & \\ \quad = e^{a(t_1-\tau)} + \displaystyle\sum_{i=1}^{2} \frac{b^i}{i!}\left(t_1-\tau-ih\right)e^{a(t_1-\tau-h)}, \text{ for } \tau \in K_2; & (10) \\ S_3 = S_2 - \displaystyle\int_{t_1-3h}^{\tau}\int_{t_1-2h}^{s_3+h}\int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-h-s_2)} A_1 e^{A_0(s_2-h-s_3)} A_1 e^{A_0(s_3-\tau)} ds_1 ds_2 ds_3 & \\ \quad = e^{a(t_1-\tau)} + \displaystyle\sum_{i=1}^{3} \frac{b^i}{i!}\left(t_1-\tau-ih\right)e^{a(t_1-\tau-h)}, \text{ for } \tau \in K_3. & (11) \end{cases}$$

3. RESULTS AND DISCUSSIONS

From the foregoing emerging pattern we state as follows:

3.1 Theorem on Control index matrices on arbitrary intervals of length h***:***

$$X(\tau,t_1) = \begin{cases} e^{a(t_1-\tau)},\ \ \tau \in K_0 & (12) \\ e^{a(t_1-\tau)} + \displaystyle\sum_{i=1}^{j} \frac{b^i}{i!}\left(t_1-\tau-ih\right)^i e^{a\left(t_1-\tau-ih\right)},\ \tau \in K_j,\ j \ge 1 : t_1-(j+1)h \ge 0. & (13) \end{cases}$$

Proof

The proof is by inductive reasoning, using backward continuation recursive approach. Obviously the theorem is valid for $j \in \{0,1,2,3\}$. Assume the validity of the theorem for $3 \le p \le j$, for some integer $j \ge 4$. Then on

$K_{j+1}, \tau + h \in K_j$ and $t_1 - [j+1]h \in K_j$. We know from (1) that:

$$X(\tau,t_1)e^{A_0(\tau-t_1)} - X(t_1-[j+1]h,\, t_1)e^{-jA_0 h} = -\int_{t_1-[j+1]h}^{\tau} X(s_{j+1}+h, A_1 e^{A_0(s_{j+1}-t_1)} ds_{j+1} \qquad (14)$$

$$\Rightarrow\ X(\tau,t_1) = X(t_1-[j+1]h,\, t_1)e^{A_0(t_1-[j+1]h-\tau)} - \int_{t_1-[j+1]h}^{\tau} X(s_{j+1}+h,\, t_1)A_1 e^{A_0(s_{j+1}-\tau)} ds_{j+1} \qquad (15)$$

Using the induction hypothesis, apply $X(\tau+h,\tau)$ and $X(t_1 \text{-} [j+1]h, t_1)$ to (13) to get:

$$X(s+h,t_1) = e^{a(t_1-h-s)} + \sum_{i=1}^{j} \frac{b^i}{i!}\left(t_1-[s+(i+1)h]\right)^i e^{a\left(t_1-[s+(i+1)h]\right)} \qquad (16)$$

$$X(t_1-[j+1]h,t_1) = e^{a(j+1)h} + \sum_{i=1}^{j} \frac{b^i}{i!}\left([j+1-i]h]\right)^i e^{a([j+1-i]h])} \qquad (17)$$

Hence:

$$X(\tau,t_1)= e^{a(t_1-\tau)h} + \sum_{i=1}^{j}\frac{b^i}{i!}([j+1-i]h])^i e^{a(t_1-\tau-ih)} - \int_{t_1-[j+1]h}^{\tau} be^{a(t_1-h-s)}e^{a(s-\tau)}ds$$

$$- \int_{t_1-[j+1]h}^{\tau} \sum_{i=1}^{j}\frac{b^{i+1}}{i!}(t_1-[s+(i+1)h])^i e^{a(t_1-\tau-ih)} \tag{18}$$

$$= e^{a(t_1-\tau)h} + \sum_{i=1}^{j}\frac{b^i}{i!}([j+1-i]h])^i e^{a(t_1-\tau-ih)} + b\left(t_1-[\tau+(j+1)h]\right)e^{a(t_1-\tau-h)}$$

$$+ \sum_{i=1}^{j}\frac{b^{i+1}}{(i+1)!}(t_1-\tau-[i+1]h])^{i+1} e^{a(t_1-\tau-[i+1]h)} - \sum_{i=1}^{j}\frac{b^{i+1}}{(i+1)!}([j-i]h])^{i+1} e^{a(t_1-\tau-[i+1]h)} \tag{19}$$

Using the change of variables $i \to i-1,$ in the summands and $i \to i+1,$ in the summation limits , we obtain:

$$X(\tau,t_1)= e^{a(t_1-\tau)h} + \sum_{i=2}^{j+1}\frac{b^i}{i!}(t_1-\tau-ih)^i e^{a(t_1-\tau-ih)} + b\left(t_1-\tau-h\right)e^{a(t_1-\tau-h)} - bjhe^{a(t_1-\tau-h)}$$

$$+ \sum_{i=1}^{j}\frac{b^i}{i!}([j+1-i]h])^i e^{a(t_1-\tau-ih)} - \sum_{i=2}^{j}\frac{b^{i+1}}{i!}([j+1-i]h])^i e^{a(t_1-\tau-ih)}$$

$$= e^{a(t_1-\tau)h} + \sum_{i=1}^{j+1}\frac{b^i}{i!}(t_1-\tau-ih)^i e^{a(t_1-\tau-ih)} + b\left(t_1-\tau-ih\right)e^{a(t_1-\tau-ih)} - b(jh)^i e^{a(t_1-\tau-h)}$$

$$+ b(jh)^i e^{a(t_1-\tau-h)} - \frac{b^{j+1}}{(j+1)!}0^i e^{a(t_1-\tau-ih)} \text{ (using } i=1 \text{ and } i=j+1 \text{ respectively)}$$

$$\Rightarrow X(\tau,t_1)=e^{a(t_1-\tau)} + \sum_{i=1}^{j+1}\frac{b^i}{i!}\left(t_1-\tau-ih\right)^i e^{a(t_1-\tau-ih)}, \text{ for } \tau\in K_{j+1}. \tag{20}$$

Therefore the theorem is valid on $K_j \ \forall j : t_1-[j+1]h \geq 0.$ This completes the proof.

4. CONCLUSION

This paper has effectively eliminated the drudgery and error-prone step-wise method of computing the indices of control systems for single- delays autonomous linear differential equations up to a desired interval of length equal to the delay, starting from the right-most interval. By obtaining their structure on arbitrary intervals, the indices of control systems for the afore-mentioned differential equations can be computed right- off- the bat on any desired interval, circumventing the afore-mentioned difficulties and providing a ready-made indispensible tool for the prosecution of the variation of constants formula for terminal function problems and the determination of the controllability dispositions of relevant differential equations. The result has wide-ranging implications for extension to systems.

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